The Vortex Lattice Grid Method for the Calculation of Flow around Wings of Arbitrary Shape and Infinite Thin Thickness

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Vortex Lattice Grid methods are based on the general idea to distribute bound and trailing vortices on the sail surface. While bound vortices produce lift, trailing vortices account for three dimensional flow around the wing. According to *Helmholtz´s* law vortice filaments must either be closed or end at the boundary of the flow domain. In order to produce lift, *Kutta´s* law must be satisfied at the leech of the sail, *Schlichting and Truckenbrodt* [1].

1 Panelization of sail

In the method applied here, the sail is discretized using any combination of triangular and quadrilateral panels. Panels may be distributed on more than one sail or wing. The sails may overlap. Fig. 1-1 shows a structured panelization of mainsail and jib of a modern racing rig, Fig. 1-2 an unstructured panelization of an Americas Cup V4 yacht.

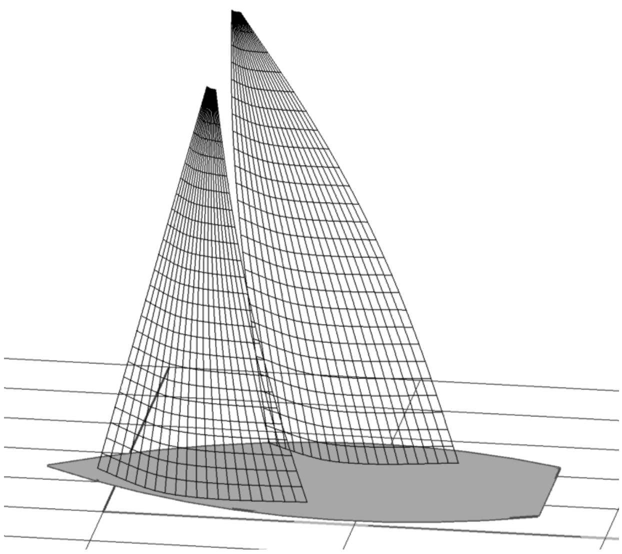

Fig. 1-1: Panelization of mainsail and jib using quadrilateral panels

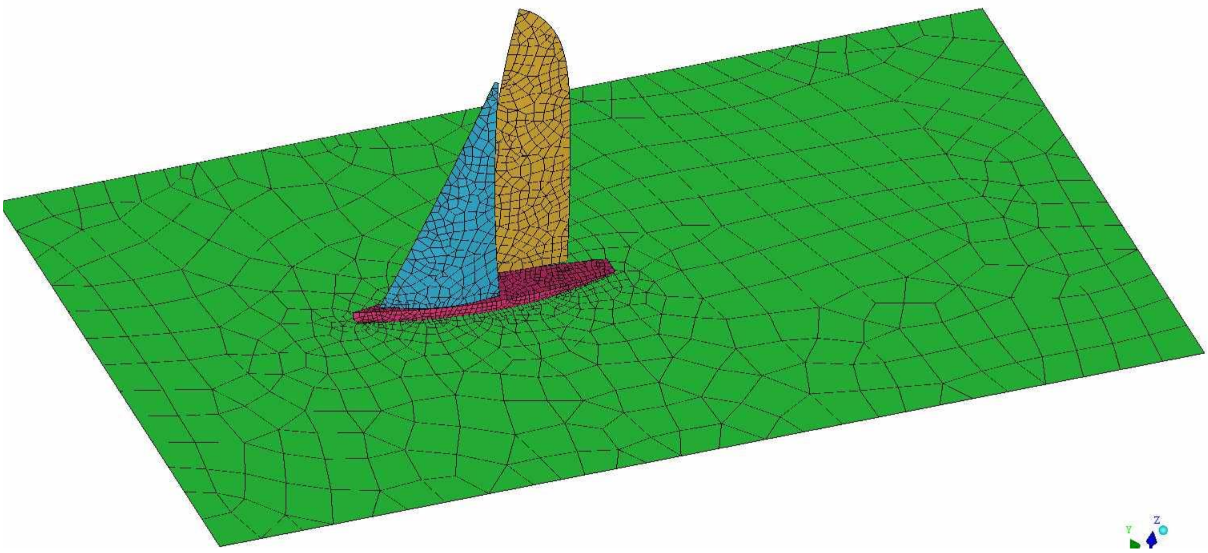

Fig. 1-2: Panelization of AC V4 yacht including mainsail, jib, canoe body and waterplane

A closed vortex filament of constant vorticity Γ_i is arranged in each panel i. At any trailing edge of the wing an additional horseshoe vortex filament is located, having same constant vorticity as neighbouring panel, however with negative orientation.

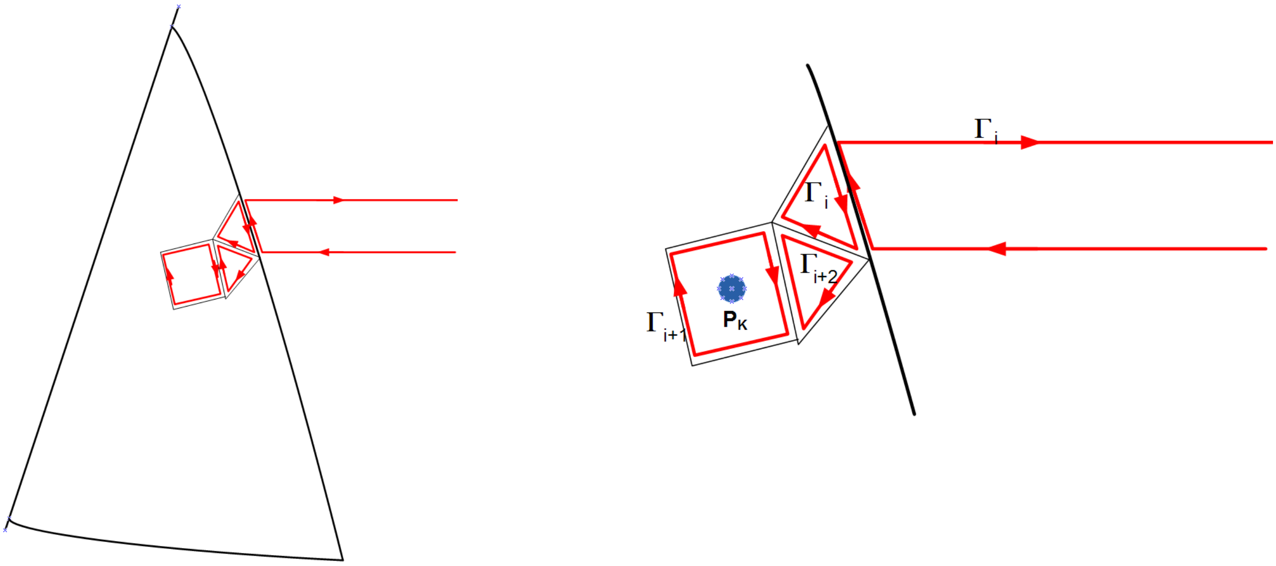

Fig. 1-3: Ring and horseshoe vortices

In the centre of each panel a collocation point $\mathbf{P}_K$ is arranged. All vortices are arranged at the exact geometric position of the panel dicretizing the wing. Thus sails of arbitrary shape, camper ratio and position, as well as sail twist and interaction of mainsail and jib are modelled accurately.

2 Calculation of vorticity

For details of the following derivation, see f. ex. *Juergens* [2].

Ring and horseshoe vortices induce velocity, which can be calculated using *Biot-Savart* law. For a vortex filament s of constant vorticity Γ the velocity induced at an arbitrary point $\mathbf{P}_K$ is calculated from:

$$\mathbf{v} = \frac{-\Gamma}{4\pi}\int_s \frac{\mathbf{r}\times d\mathbf{s}}{|\mathbf{r}|^3} \qquad (2.1)$$

where $\mathbf{\Gamma}$ is the vorticity and $\mathbf{r}$ is a vector from the position $\mathbf{P}_K$ where the induced velocity is calculated to the infinite vector along the vortex filament $d\mathbf{s}$. The integration is carried out over the entire vortex lines s.

For a straight vortex filament we use the plane defined by the filament and $\mathbf{P}_K$ to solve (2.1). $\mathbf{v}$ is directed perpendicular to this plane. The magnitude of $\mathbf{v}$:

$$v = \frac{-\Gamma}{4\pi}\int_s \frac{|\mathbf{r}\times d\mathbf{s}|}{|\mathbf{r}|^3}$$

can be calculated as follows, see Fig. 2-1. We are using a polar coordinate system *r,φ,* it's origin located at the point where the induced velocity will be calculated. A vector (*r,0)* points perpendicular to the vortex filament. We then deduce:

$$|\mathbf{r}\times d\mathbf{s}| = r\,ds\sin\beta = r\,ds\cos\varphi$$

For *ds* we derive $ds = \dfrac{r\,d\varphi}{\cos\varphi}$ for an infinite small *dφ.*

Consequently $\dfrac{|\mathbf{r}\times d\mathbf{s}|}{|\mathbf{r}|^3} = \dfrac{d\varphi}{r} = \dfrac{\cos\varphi}{a}d\varphi$ using $r = a/\cos\varphi$.

a is a constant for any integration over the straight vortex filament. We thus get:

$$v = \frac{-\Gamma}{4\pi}\int_s \frac{|\mathbf{r}\times d\mathbf{s}|}{|\mathbf{r}|^3} = \frac{-\Gamma}{4\pi a}\int_s \cos\varphi\,d\varphi = \frac{-\Gamma}{4\pi a}(\sin\varphi_1 + \sin\varphi_2) \qquad = \frac{\Gamma}{4\pi a}(\cos\alpha + \cos\gamma)$$

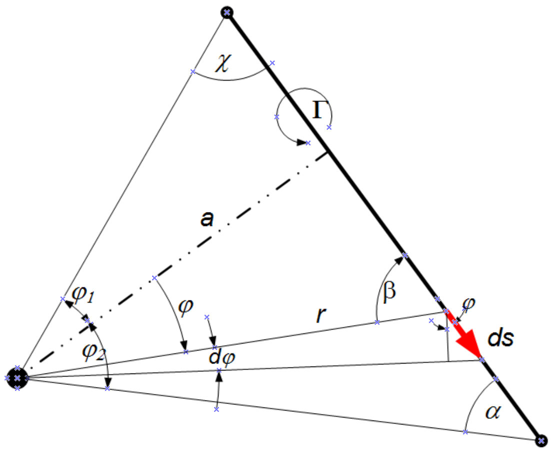

Fig. 2-1: Induction of velocitiy due to straight vortex filament

For an arbitrary location of a straight vortex filament, defined by the points **P** $\mathbf{P_A}$ and $\mathbf{P_B}$ and an arbitrary point $\mathbf{P_K}$ where **v** shall be calculated we use the following expressions:

An arbitrary located single straight vortex filament of constant vorticity Γ, which runs from point $\mathbf{P}_A$ to $\mathbf{P}_B$, , see Fig. 2-2, induces a velocity $\mathbf{v}_s$ in point $\mathbf{P}_K$, which can be calculated from:

$$\mathbf{v}_s = |\mathbf{v}_s|\ \mathbf{n}_K$$

where

$$|\mathbf{v}_S| = \frac{\Gamma}{4\pi a}(\cos\alpha + \cos\gamma)$$

is the magnitude of the induced velocity and

$$\mathbf{n}_K = \frac{\mathbf{r}_A \times \mathbf{r}_B}{|\mathbf{r}_A \times \mathbf{r}_B|}$$

a unity vector in the direction of the induced velocity. $\mathbf{n}_K$ is oriented perpendicular on the triangle defined by $\mathbf{P}_A$, $\mathbf{P}_B$ and $\mathbf{P}_K$. $|\mathbf{v}_S|$ is calculated using the following quantities:

$$\cos\alpha = \frac{\mathbf{s}_{AB}\cdot\mathbf{r}_A}{|\mathbf{s}_{AB}||\mathbf{r}_A|}, \quad \cos\gamma = \frac{-\mathbf{s}_{AB}\cdot\mathbf{r}_B}{|\mathbf{s}_{AB}||\mathbf{r}_B|}, \quad a = |\mathbf{r}_A|\sqrt{1-\cos^2\alpha}\ ,$$

The vectors describing the triangle are

$$\mathbf{r}_A = \mathbf{P}_K - \mathbf{P}_A, \quad \mathbf{r}_B = \mathbf{P}_K - \mathbf{P}_B \quad \textbf{and} \quad \mathbf{s}_{AB} = \mathbf{P}_B - \mathbf{P}_A$$

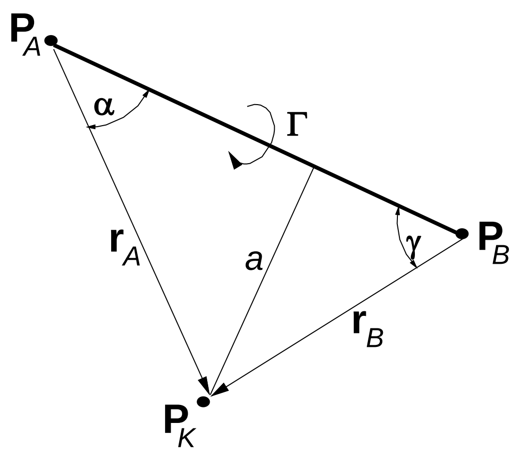

Fig. 2-2: Straight vortex line

The vorticities Γ are calculated using the boundary condition on the sail surface: the sum of induced velocity $\mathbf{v}$ of all vortex rings and horseshoe vortices and the wind velocity $\mathbf{u}_{Wind}$ has no component normal to the sail surface:

$$(\mathbf{u}_{Wind} + \mathbf{v})\cdot\mathbf{n}_S = 0$$

$\mathbf{n}_S$ is a the normal unity vector on the sail surface. This boundary condition is established in the collocation point of each panel $\mathbf{P}_K$. Applying this boundary condition to the collocation points in all panels the following linear system of equations arises:

$$\sum m_{ij}\Gamma_j = -\mathbf{u}_i\cdot\mathbf{n}_i$$

Here Γ_j denotes the vorticity of the vortex ring or vortex horseshoe in panel j, $\mathbf{u}_i$ and $\mathbf{n}_i$ denote the wind velocity and the normal unity vector in collocation point i and

$$m_{ij} = \frac{1}{\Gamma_j}\mathbf{v}_{ij}\cdot\mathbf{n}_i$$

$\mathbf{v}_{ij}$ is the velocity induced by the horseshoe or ring vortex j at collocation point i. $\mathbf{v}_{ij}$ is calculated from summation of the velocities induced by the four (three) straight vortex filaments which form a vortex ring or horseshoe vortex.

$$m_{ij} = \sum_{filaments}\frac{1}{4\pi\, a_{ij}}(\cos\alpha_{ij} + \cos\chi_{ij})\mathbf{n}_{ij}\mathbf{n}_i$$

Here a_{ij} is the distance from the filament of horseshoe or ring vortex j to collocation point i, $\mathbf{n}_{ij}$ is the unity vector normal to the triangle described by the current filament of horseshoe or ring vortex j to collocation point i and α_{ij} and χ_{ij} are the respective angles of the triangle opposed to the collocation point i:

$$a_{ij} = \left|\mathbf{r}_{Aij}\right|\sqrt{1-\cos^2\alpha_{ij}}$$

$$\cos\alpha_{ij} = \frac{\mathbf{s}_{AB\,j}\cdot\mathbf{r}_{Aij}}{\left|\mathbf{s}_{AB\,j}\right|\left|\mathbf{r}_{Aij}\right|} \qquad \cos\chi_{ij} = \frac{-\mathbf{s}_{AB\,j}\cdot\mathbf{r}_{Bij}}{\left|\mathbf{s}_{AB\,j}\right|\left|\mathbf{r}_{Bij}\right|}$$

$$\mathbf{n}_{ij} = \frac{\mathbf{r}_{Aij}\times\mathbf{r}_{Bij}}{\left|\mathbf{r}_{Aij}\times\mathbf{r}_{Bij}\right|}$$

Here $\mathbf{r}_{Aij}$ is the vector from the start of the current filament of the horseshoe or ring vortex j to the collocation point i:

$$\mathbf{r}_{Aij} = \mathbf{P}_{K\,i} - \mathbf{P}_{A\,j}$$

while $\mathbf{r}_{Bij}$ is the vector from the second end of the current filament of the horseshoe or ring vortex j to the collocation point i:

$$\mathbf{r}_{Bij} = \mathbf{P}_{K\,i} - \mathbf{P}_{B\,j}$$

$\mathbf{s}_{AB\,j}$ is the vector from the start to the end of the current filament of horseshoe or ring vortex j:

$$\mathbf{s}_{AB\,j} = \mathbf{P}_{B\,j} - \mathbf{P}_{A\,j}$$

3 Calculation of sail forces

After calculation of the discrete vorticities Γ the force on a particular straight vortex line *k* of the ring vortex j is calculated by

$$\mathbf{F}_{jk} = \rho\Gamma_j\,(\mathbf{u}_{Wind} + \hat{\mathbf{v}})\times(\mathbf{s}_{AB})$$

Here ρ denotes the density of the fluid, Γ_j the vorticity of the straight vortex line k of the panel j, $\mathbf{u}_{Wind}$ the wind velocity, $\mathbf{s}_{AB}$ the vector along the vortex filament and $\hat{\mathbf{v}}$ the velocity induced in the centre of the straight vortex filament by all horseshoe and ring vortices discretizing the sail with the exception of the vortex filament of which the actual forces are calculated (no self-induction). $\hat{\mathbf{v}}$ can be calculated from:

$$\hat{\mathbf{v}} = \sum_j \sum_{filaments} \frac{\Gamma_j}{4\pi a_{ij}}(\cos\alpha_{ij} + \cos\chi_{ij})\frac{\mathbf{n}_{ij}}{\left|\mathbf{n}_{ij}\right|}$$

Distance *a* and angles α and χ are calculated as above, however instead of the collocation point *i* the centre of the filament has to be taken for the calculation of the radii $\mathbf{r_A}$ and $\mathbf{r}_B$. To avoid self-induction, the contribution to the induced velocity must be set to zero for a=0.

The moment **M** is calculated by

$$\mathbf{M}_{jk} = \mathbf{r}_{jk} \times \mathbf{F}_{jk}$$

where $\mathbf{r}_{jk}$ is a vector to the centre of the filament k of the vortex ring j.

Forces and moments on the complete sail are calculated by summing the discrete forces and moment for all filaments of any ring and horseshoe vortex.

1 Schlichting, H. and Truckenbrodt, E.: *Aerodynamic des Flugzeuges*, Vol. 1, Springer Verlag, Berlin, 1967

2 Juergens, D.: *Theoretische und experimentelle Untersuchung instationärer Tragflügelumströmungen und Entwicklung eines Berechnungsverfahrens für Vertikalachsenrotoren*, Ph.D. thesis, Univ. Rostock, 1994